

BOUNDS FOR THE DIFFERENCE BETWEEN TWO ČEBYŠEV FUNCTIONALS

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ABSTRACT. In this work, a generalization of pre-Grüss inequality is established. Several bounds for the difference between two Čebyšev functional are proved.

1. INTRODUCTION

It is well known that for a continuous function f defined on $[a, b]$, the integral mean-value theorem (IMVT) guarantees $x \in [a, b]$ such that

$$(1.1) \quad f(x) = \frac{1}{b-a} \int_a^b f(t) dt.$$

On the other hand, for a monotonic function $g : [a, b] \rightarrow \mathbb{R}$ that does not change sign in the interval $[a, b]$, the weighted IMVT reads that there exists $x \in [a, b]$ such that

$$(1.2) \quad \int_a^b f(t) g(t) dt = f(x) \int_a^b g(t) dt.$$

If one replaces the value of $f(x)$ in (1.2) by its value in (1.1) then we get

$$(1.3) \quad \int_a^b f(t) g(t) dt = \frac{1}{b-a} \int_a^b f(t) dt \int_a^b g(t) dt.$$

To get weighted values in (1.3) we divide the both sides by the quantity ‘ $b-a$ ’ to get

$$(1.4) \quad \frac{1}{b-a} \int_a^b f(t) g(t) dt = \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt,$$

which means in such way that the weighted product of two functions equal to the product of weights of that functions.

The difference between these weights

$$(1.5) \quad \mathcal{T}_a^b(f, g) = \frac{1}{b-a} \int_a^b f(t) g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt.$$

is called ‘the Čebyšev functional’, which plays an important role in Numerical Approximations and Operator Theory. For more detailed history see [17].

The most famous bounds for the Čebyšev functional are incorporated in the following theorem:

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Theorem 1. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two absolutely continuous functions, then*

(1.6)

$$|\mathcal{T}_a^b(f, g)| \leq \begin{cases} \frac{(b-a)^2}{12} \|f'\|_\infty \|g'\|_\infty, & \text{if } f', g' \in L_\infty[a, b], \quad \text{proved in [11]} \\ \frac{1}{4} (M_1 - m_1) (M_2 - m_2), & \text{if } m_1 \leq f \leq M_1, \quad m_2 \leq g \leq M_2, \quad \text{proved in [14]} \\ \frac{(b-a)}{\pi^2} \|f'\|_2 \|g'\|_2, & \text{if } f', g' \in L_2[a, b], \quad \text{proved in [16]} \\ \frac{1}{8} (b-a) (M - m) \|g'\|_\infty, & \text{if } m \leq f \leq M, \quad g' \in L_\infty[a, b], \quad \text{proved in [18]} \end{cases}$$

The constants $\frac{1}{12}$, $\frac{1}{4}$, $\frac{1}{\pi^2}$ and $\frac{1}{8}$ are the best possible.

Many authors were studied the functional (1.5) and therefore various bounds have been implemented, for more new results and generalizations the reader may refer to [1],[2],[6],[7],[9],[12],[15] and [19].

In 2001, Cerone [10] established the following identity for the Čebyšev functional:

Theorem 1. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be such that f is of bounded variation and g is continuous on $[a, b]$. Then, we have the following representation:*

$$(1.7) \quad \mathcal{T}_a^b(f, g) = \frac{1}{(b-a)^2} \int_a^b \left[(t-a) \int_t^b g(s) ds - (b-t) \int_a^t g(s) ds \right] df(t).$$

In 2007, Dragomir [13] established three equivalent identities that generalized Cerone identity (1.7) for Riemann-Stieltjes integrals, in case of Riemann integral Dragomir representation incorporated in the following theorem.

Theorem 2. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be such that f is of bounded variation and g is Lebesgue integrable on $[a, b]$. Then,*

$$(1.8) \quad \mathcal{T}_a^b(f, g) = \frac{1}{(b-a)^2} \int_a^b \left[(t-a) \int_a^b g(t) dt - (b-a) \int_a^t g(s) ds \right] df(t).$$

The absolute difference between two integral means was studied firstly by Barnett et al. in [5] and then by Cerone and Dragomir in [8], we may summarize the obtained results, as follow:

• For an absolutely continuous function f defined on $[a, b]$ and for all $a \leq c < d \leq b$, we have

$$(1.9) \quad \begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{d-c} \int_c^d f(s) ds \right| \\ & \leq \left[\frac{1}{4} + \left(\frac{(a+b)/2 - (c+d)/2}{(b-a) - (d-c)} \right)^2 \right] [(b-a) - (d-c)] \|f'\|_\infty \\ & \leq \frac{1}{2} [(b-a) - (d-c)] \|f'\|_\infty \end{aligned}$$

and

$$(1.10) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{d-c} \int_c^d f(s) ds \right| \leq \begin{cases} \frac{(b-a)}{(q+1)^{1/q}} \left[1 + \left(\frac{\rho}{1-\rho} \right)^q \right]^{1/q} [v^{q+1} + \lambda^{q+1}]^{1/q} \|f'\|_p, \\ f' \in L_p[a, b], \quad 1 \leq p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} [1 - \rho + |v - \lambda|] \|f'\|_1, \quad f' \in L_1[a, b]; \end{cases}$$

where $(b-a)v = c-a$, $(b-a)\rho = d-c$ and $(b-a)\lambda = b-d$.

• For a Hölder continuous function f of order $r \in (0, 1]$ with constant $H > 0$ on $[a, b]$, we have

$$(1.11) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{d-c} \int_c^d f(s) ds \right| \leq H \frac{(c-a)^{r+1} + (b-d)^{r+1}}{(r+1)[(b-a) - (d-c)]}.$$

• For a function f of bounded variation on $[a, b]$, we have

$$(1.12) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{d-c} \int_c^d f(s) ds \right| \leq \begin{cases} \left[\frac{b-a-(d-c)}{2} + \left| \frac{c+d}{2} - \frac{a+b}{2} \right| \right] \frac{V_a^b(f)}{b-a}; \\ L \frac{(c-a)^2 + (b-d)^2}{2[(b-a) - (d-c)]}; \quad \text{if } f \text{ is L-Lipschitzian} \\ \left(\frac{b-d}{b-a} \right) f(b) - \left(\frac{c-a}{b-a} \right) f(a) + \left[\frac{c+d-(a+b)}{b-a} \right] f(s_0); \\ \text{if } f \text{ is monotonic nondecreasing} \end{cases}$$

where, $s_0 = \frac{cb-ad}{(b-a)-(d-c)} \in [c, d]$.

For recent results the reader may refer to [3], where the author used (1.8) to obtain several bounds for the Čebyšev functional. Bounds for the difference between two Stieltjes integral means was presented in [4].

Let $g : [\alpha, \beta] \rightarrow \mathbb{R}$ be any integrable function and define $\Psi : [\alpha, \beta] \rightarrow \mathbb{R}$, such that

$$\Psi_g(t; \alpha, \beta) := \int_{\alpha}^t g(s) ds - \frac{t-\alpha}{\beta-\alpha} \int_{\alpha}^{\beta} g(s) ds.$$

From (1.8), it is easy to observe the following representation of the Čebyšev functional

$$\mathcal{T}_{\alpha}^{\beta}(f, g) := -\frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \Psi_g(t; \alpha, \beta) df(t).$$

In this work by utilizing the inequalities (1.9)–(1.12), several new bounds for the absolute *Difference between two Čebyšev functional* $\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)$, for all $a \leq u < v \leq b$ are provided.

Let us start by providing the following refinements of pre-Grüss inequality, which states that for any two integrable mappings defined on $[a, b]$, the inequality

$$(1.13) \quad \mathcal{T}_a^b(f, g) \leq [\mathcal{T}_a^b(f, f)]^{1/2} \cdot [\mathcal{T}_a^b(g, g)]^{1/2},$$

holds and sharp (see [14]). Trivially, by applying AM–GM inequality on the right hand side of (1.13), we get

$$(1.14) \quad [\mathcal{T}_a^b(f, f)]^{1/2} \cdot [\mathcal{T}_a^b(g, g)]^{1/2} \leq \frac{\mathcal{T}_a^b(f, f) + \mathcal{T}_a^b(g, g)}{2}.$$

We may generalize the pre-Grüss inequality (1.13) as follows:

Theorem 3. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two integrable mappings, then*

$$(1.15) \quad \begin{aligned} & |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \\ & \leq (\mathcal{T}_a^v(f, f))^{1/2} (\mathcal{T}_a^v(g, g))^{1/2} + (\mathcal{T}_u^b(f, f))^{1/2} (\mathcal{T}_u^b(g, g))^{1/2} \\ & \leq \frac{1}{2} [\mathcal{T}_a^v(f, f) + \mathcal{T}_a^v(g, g) + \mathcal{T}_u^b(f, f) + \mathcal{T}_u^b(g, g)], \end{aligned}$$

for all $a \leq u < v \leq b$. The double inequality is sharp.

Proof. Simply using the (1.13), we have

$$\begin{aligned} & |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)|^2 \\ & \leq (\mathcal{T}_a^v(f, g))^2 + 2\mathcal{T}_a^v(f, g) \cdot \mathcal{T}_u^b(f, g) + (\mathcal{T}_u^b(f, g))^2 \\ & \leq (\mathcal{T}_a^v(f, f)) (\mathcal{T}_a^v(g, g)) + 2\mathcal{T}_a^v(f, g) \cdot \mathcal{T}_u^b(f, g) + (\mathcal{T}_u^b(f, f)) (\mathcal{T}_u^b(g, g)) \\ & = \left[(\mathcal{T}_a^v(f, f))^{1/2} (\mathcal{T}_a^v(g, g))^{1/2} + (\mathcal{T}_u^b(f, f))^{1/2} (\mathcal{T}_u^b(g, g))^{1/2} \right] \\ & \quad \times (\mathcal{T}_a^v(f, f))^{1/2} (\mathcal{T}_a^v(g, g))^{1/2} \\ & \quad + \left[(\mathcal{T}_u^b(f, f))^{1/2} (\mathcal{T}_u^b(g, g))^{1/2} + (\mathcal{T}_a^v(f, f))^{1/2} (\mathcal{T}_a^v(g, g))^{1/2} \right] \\ & \quad \times (\mathcal{T}_u^b(f, f))^{1/2} (\mathcal{T}_u^b(g, g))^{1/2} \\ & = \left[(\mathcal{T}_a^v(f, f))^{1/2} (\mathcal{T}_a^v(g, g))^{1/2} + (\mathcal{T}_u^b(f, f))^{1/2} (\mathcal{T}_u^b(g, g))^{1/2} \right]^2 \end{aligned}$$

and this implies the first inequality in (1.15). The second inequality follows by applying the AM–GM inequality. The sharpness follows by letting $f = g = x$. \square

Remark 1. *We note that (1.15) reduces to (1.13) by setting $u = a$ and $v = u + \epsilon$, thus*

$$|\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \longrightarrow |\mathcal{T}_a^b(f, g)| \quad \text{as } \epsilon \longrightarrow 0^+.$$

Consequently, the right hand of (1.15) \longrightarrow the right hand of (1.13).

2. BOUNDS FOR BOUNDED VARIATION INTEGRATORS

The first result regarding bounded variation integrators is presented as follows:

Theorem 4. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be such that f is of bounded variation on $[a, b]$ and g is absolutely continuous on $[a, b]$, then

$$(2.1) \quad |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \leq \bigvee_a^b(f) \cdot \begin{cases} \frac{1}{8} \left[\frac{(v-a)+(b-u)}{2} + \left| \frac{b-u}{2} - \frac{v-a}{2} \right| \right] \|g'\|_{\infty, [a, b]}, & \text{if } g' \in L_\infty[a, b]; \\ \frac{1}{2(q+1)^{1/q}} \left[\frac{b-a}{2} + \left| v - \frac{a+b}{2} \right| \right] \cdot \|g'\|_{p, [a, b]}, & \text{if } g' \in L_p[a, b], \\ \frac{1}{2} \|g'\|_{1, [a, b]}, & \text{if } g' \in L_1[a, b], \end{cases},$$

for all $a \leq u < v \leq b$, where $\|\cdot\|_p$ are the usual Lebesgue norms, i.e.,

$$\|h\|_p := \left(\int_a^b |h(t)|^p dt \right)^{1/p}, \text{ for } p \geq 1$$

and

$$\|h\|_\infty := \operatorname{ess\,sup}_{t \in [a, b]} |h(t)|.$$

Proof. It is known that for a continuous function w on $[a, b]$ and a bounded variation ν on $[a, b]$, one have the inequality

$$(2.2) \quad \left| \int_a^b w(t) d\nu(t) \right| \leq \sup_{t \in [a, b]} |w(t)| \bigvee_a^b(\nu).$$

Employing (2.2) for the Cerone-Dragomir identity

$$(2.3) \quad \mathcal{T}(f, g) = -\frac{1}{b-a} \int_a^b \left(\int_a^t g(s) ds - \frac{t-a}{b-a} \int_a^b g(s) ds \right) df(t).$$

One has as f is of bounded variation on $[a, b]$,

$$(2.4) \quad \begin{aligned} & |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \\ &= \left| \frac{1}{v-a} \int_a^v \left(\int_a^t g(s) ds - \frac{t-a}{v-a} \int_a^v g(s) ds \right) df(t) \right. \\ &\quad \left. - \frac{1}{b-u} \int_u^b \left(\int_u^r g(s) ds - \frac{r-u}{b-u} \int_u^b g(s) ds \right) df(t) \right| \\ &\leq \left| \frac{1}{v-a} \int_a^v \left(\int_a^t g(s) ds - \frac{t-a}{v-a} \int_a^v g(s) ds \right) df(t) \right| \\ &\quad + \left| \frac{1}{b-u} \int_u^b \left(\int_u^r g(s) ds - \frac{r-u}{b-u} \int_u^b g(s) ds \right) df(t) \right| \\ &\leq \frac{1}{v-a} \sup_{t \in [a, v]} \left| \int_a^t g(s) ds - \frac{t-a}{v-a} \int_a^v g(s) ds \right| dt \cdot \bigvee_a^v(f) \\ &\quad + \frac{1}{b-u} \sup_{r \in [u, b]} \left| \int_u^r g(s) ds - \frac{r-u}{b-u} \int_u^b g(s) ds \right| dt \cdot \bigvee_u^b(f) \end{aligned}$$

In the inequality (1.9), setting $d = t$, $c = a$ and then $d = r$, $c = u$, we get

$$(2.5) \quad \left| \frac{1}{t-a} \int_a^t g(s) ds - \frac{1}{v-a} \int_a^v g(s) ds \right| \leq \frac{1}{2} (v-t) \|g'\|_{\infty, [a, v]}$$

and

$$(2.6) \quad \left| \frac{1}{r-u} \int_u^r g(s) ds - \frac{1}{b-u} \int_u^b g(s) ds \right| \leq \frac{1}{2} (b-r) \|g'\|_{\infty, [u, b]}.$$

Substituting (2.5) and (2.6) in (2.4), we get

$$(2.7) \quad \begin{aligned} |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| &\leq \frac{1}{v-a} \cdot \frac{1}{2} \|g'\|_{\infty, [a, v]} \sup_{t \in [a, v]} \{(t-a)(v-t)\} \bigvee_a^v(f) \\ &\quad + \frac{1}{b-u} \cdot \frac{1}{2} \|g'\|_{\infty, [u, b]} \sup_{r \in [u, b]} \{(r-u)(b-r)\} \bigvee_u^b(f) \\ &= \frac{1}{8} (v-a) \|g'\|_{\infty, [a, v]} \bigvee_a^v(f) + \frac{1}{8} (b-u) \|g'\|_{\infty, [u, b]} \bigvee_u^b(f) \\ &\leq \frac{1}{8} \max\{(v-a), (b-u)\} \|g'\|_{\infty, [a, b]} \bigvee_a^b(f) \\ &\leq \frac{1}{8} \left[\frac{(v-a) + (b-u)}{2} + \left| \frac{b-u}{2} - \frac{v-a}{2} \right| \right] \|g'\|_{\infty, [a, b]} \bigvee_a^b(f) \end{aligned}$$

where we used the fact that $\sup_{t \in [\alpha, \beta]} \{(t-\alpha)(\beta-t)\}$, occurs at $t = \frac{\alpha+\beta}{2}$, therefore,

$\sup_{t \in [\alpha, \beta]} \{(t-\alpha)(\beta-t)\} = \frac{1}{4} (\beta-\alpha)^2$. Also, we note that the last inequality holds since

$$\|g'\|_{\infty, [a, v]} \leq \|g'\|_{\infty, [a, b]}, \quad \bigvee_a^v(f) \leq \bigvee_a^b(f) \quad \text{and} \quad \bigvee_u^b(f) \leq \bigvee_a^b(f),$$

which proves the first inequality in (2.1).

In the inequality (1.10), replace r, u instead of d, c ; respectively and then t, a instead of d, c ; respectively, we find that

$$(2.8) \quad \begin{aligned} &\left| \frac{1}{r-a} \int_a^r g(s) ds - \frac{1}{v-a} \int_a^v g(s) ds \right| \\ &\leq \begin{cases} \frac{(v-r)^{\frac{1}{q}}}{(q+1)^{1/q} (v-a)^{\frac{1}{q}}} [(r-a)^q + (v-r)^q]^{1/q} \|g'\|_{p, [a, v]}, & g' \in L_p[a, v], \\ \frac{v-r}{v-a} \|g'\|_{1, [a, v]}, & g' \in L_1[a, v]. \end{cases} \end{aligned}$$

and

$$(2.9) \quad \left| \frac{1}{t-u} \int_u^t g(s) ds - \frac{1}{b-a} \int_u^b g(s) ds \right| \leq \begin{cases} \frac{(b-t)^{\frac{1}{q}}}{(q+1)^{1/q}(b-u)^{\frac{1}{q}}} [(t-u)^q + (b-t)^q]^{1/q} \|g'\|_{p,[u,b]}, & g' \in L_p[u, b], \\ \frac{b-t}{b-u} \|g'\|_{1,[u,b]}, & g' \in L_1[u, b] \end{cases}$$

Substituting (2.8) and (2.9) in (2.4), we have respectively

$$\begin{aligned} & \frac{1}{v-a} \sup_{r \in [a,v]} (r-a) \left| \frac{1}{r-a} \int_a^r g(s) ds - \frac{1}{v-a} \int_a^v g(s) ds \right| \\ & \leq \frac{1}{(v-a)} \begin{cases} \frac{\|g'\|_{p,[a,v]}}{(q+1)^{1/q}(v-a)^{1/q}} \sup_{t \in [a,v]} \left\{ (r-a)(v-r)^{\frac{1}{q}} [(r-a)^q + (v-r)^q]^{1/q} \right\}, & g' \in L_p[a, v], \\ \frac{\|g'\|_{1,[a,v]}}{v-a} \sup_{r \in [a,v]} (r-a)(v-r), & g' \in L_1[a, v], \end{cases} \end{aligned}$$

$$(2.10) \quad = \begin{cases} \frac{(v-a)}{4(q+1)^{1/q}} \cdot \|g'\|_{p,[a,v]}, & g' \in L_p[a, v], \\ \frac{1}{4} \|g'\|_{1,[a,v]}, & g' \in L_1[a, v], \end{cases},$$

and similarly, we have

$$(2.11) \quad \begin{aligned} & \frac{1}{b-u} \sup_{r \in [u,b]} (r-u) \left| \frac{1}{t-u} \int_u^t g(s) ds - \frac{1}{b-u} \int_u^b g(s) ds \right| \\ & \leq \begin{cases} \frac{(b-u)}{4(q+1)^{1/q}} \cdot \|g'\|_{p,[u,b]}, & g' \in L_p[u, b], \\ \frac{1}{4} \|g'\|_{1,[u,b]}, & g' \in L_1[u, b]. \end{cases} \end{aligned}$$

Adding (2.10) and (2.11), we get

$$\begin{aligned} & |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \\ & \leq \begin{cases} \frac{(v-a)}{4(q+1)^{1/q}} \cdot \|g'\|_{p,[a,v]} V_a^v(f) + \frac{(b-u)}{4(q+1)^{1/q}} \cdot \|g'\|_{p,[u,b]} V_u^b(f), & g' \in L_p[u, b], \\ \frac{1}{4} \|g'\|_{1,[a,v]} V_a^v(f) + \frac{1}{4} \|g'\|_{1,[u,b]} V_u^b(f), & g' \in L_1[u, b], \end{cases} \\ & \leq \begin{cases} \frac{1}{2(q+1)^{1/q}} \left[\frac{b-a}{2} + \left| v - \frac{a+b}{2} \right| \right] \cdot \|g'\|_{p,[a,b]} V_a^b(f), & g' \in L_p[a, b], \\ \frac{1}{2} \|g'\|_{1,[a,b]} V_a^b(f), & g' \in L_1[a, b], \end{cases} \end{aligned}$$

which proves the second and the third inequalities in (2.1)

□

Corollary 1. *Under the assumptions of Theorem 4, we have*

$$(2.12) \quad \left| \mathcal{T}_a^u(f, g) - \mathcal{T}_u^b(f, g) \right| \leq \bigvee_a^b(f) \cdot \begin{cases} \frac{1}{8} \left[\frac{b-a}{2} + \left| u - \frac{a+b}{2} \right| \right] \|g'\|_{\infty, [a, b]}, & \text{if } g' \in L_\infty[a, b]; \\ \frac{1}{2(q+1)^{1/q}} \left[\frac{b-a}{2} + \left| u - \frac{a+b}{2} \right| \right] \|g'\|_{p, [a, b]}, & \text{if } g' \in L_p[a, b], \\ \frac{1}{2} \|g'\|_{1, [a, b]}, & \text{if } g' \in L_1[a, b]. \end{cases}$$

for all $a \leq u \leq b$. In particular case if $u = \frac{a+b}{2}$, we get

$$(2.13) \quad \left| \mathcal{T}_a^{\frac{a+b}{2}}(f, g) - \mathcal{T}_{\frac{a+b}{2}}^b(f, g) \right| \leq \bigvee_a^b(f) \cdot \begin{cases} \frac{b-a}{16} \|g'\|_{\infty, [a, b]}, & \text{if } g' \in L_\infty[a, b]; \\ \frac{b-a}{4(q+1)^{1/q}} \cdot \|g'\|_{p, [a, b]}, & \text{if } g' \in L_p[a, b], \\ \frac{1}{2} \|g'\|_{1, [a, b]}, & \text{if } g' \in L_1[a, b]. \end{cases}$$

Proof. In Theorem 4, let $\epsilon > 0$ and set $v = u + \epsilon$ so as $\epsilon \rightarrow 0^+$ we get the required result. \square

Another result when g is of r - H -Hölder type is as follows:

Theorem 5. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be such that f is of bounded variation on $[a, b]$ and g is of p - H -Hölder type on $[a, b]$, for $p \in (0, 1]$ and $H > 0$ are given. Then*

$$(2.14) \quad \left| \mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g) \right| \leq H \frac{(v-a)^p + (b-u)^p}{2^{p+1}(p+1)} \bigvee_a^b(f),$$

and

$$(2.15) \quad \left| \mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g) \right| \leq \frac{H}{2^p(p+1)} \left[\frac{(v-a) + (b-u)}{2} + \left| \frac{v-a}{2} - \frac{b-u}{2} \right| \right]^p \cdot \bigvee_a^b(f),$$

for all $a \leq u < v \leq b$.

Proof. We repeat the proof of Theorem 4. So as f is of bounded variation and g is of p - H -Hölder type on $[a, b]$, then we have

$$\begin{aligned}
& |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \\
& \leq \frac{1}{v-a} \sup_{r \in [a, v]} \left| (r-a) \left[\frac{1}{r-a} \int_a^r g(s) ds - \frac{1}{v-a} \int_a^v g(s) ds \right] \right| \bigvee_a^v(f) \\
& \quad + \frac{1}{b-u} \sup_{t \in [u, b]} \left| (t-u) \left[\frac{1}{t-u} \int_u^t g(s) ds - \frac{1}{b-u} \int_u^b g(s) ds \right] \right| \bigvee_u^b(f) \\
& \leq \frac{1}{v-a} \frac{H}{p+1} \sup_{r \in [a, v]} (r-a) (v-r)^p \bigvee_a^v(f) + \frac{1}{b-u} \frac{H}{p+1} \sup_{t \in [u, b]} (t-u) (b-t)^p \bigvee_u^b(f) \\
& = H \frac{(v-a)^p}{2^{p+1}(p+1)} \bigvee_a^v(f) + H \frac{(b-u)^p}{2^{p+1}(p+1)} \bigvee_u^b(f) \\
& \leq H \frac{(v-a)^p + (b-u)^p}{2^{p+1}(p+1)} \bigvee_a^b(f),
\end{aligned}$$

which proves the first inequality. To obtain the second inequality from the above inequality we may obtain that

$$\begin{aligned}
& |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \\
& \leq H \frac{(v-a)^p}{2^{p+1}(p+1)} \bigvee_a^v(f) + H \frac{(b-u)^p}{2^{p+1}(p+1)} \bigvee_u^b(f) \\
& \leq H \frac{1}{2^{p+1}(p+1)} \max\{(v-a)^p, (b-u)^p\} \left[\bigvee_a^v(f) + \bigvee_u^b(f) \right] \\
& \leq H \frac{1}{2^p(p+1)} \left[\frac{(v-a) + (b-u)}{2} + \left| \frac{v-a}{2} - \frac{b-u}{2} \right| \right]^p \cdot \bigvee_a^b(f).
\end{aligned}$$

which proves (2.15), and thus the proof is completed. \square

Corollary 2. Under the assumptions of Theorem 5, we have

$$(2.16) \quad |\mathcal{T}_a^u(f, g) - \mathcal{T}_u^b(f, g)| \leq H \frac{(u-a)^p + (b-u)^p}{2^{p+1}(p+1)} \bigvee_a^b(f),$$

and

$$(2.17) \quad |\mathcal{T}_a^u(f, g) - \mathcal{T}_u^b(f, g)| \leq \frac{H}{2^p(p+1)} \left[\frac{b-a}{2} + \left| u - \frac{a+b}{2} \right| \right]^p \cdot \bigvee_a^b(f),$$

for all $a \leq u \leq b$. In particular case if $u = \frac{a+b}{2}$, then the both inequalities (2.16) and (2.17) gives the same inequality, that is

$$(2.18) \quad \left| \mathcal{T}_a^{\frac{a+b}{2}}(f, g) - \mathcal{T}_{\frac{a+b}{2}}^b(f, g) \right| \leq H \frac{(b-a)^p}{2^{2p}(p+1)} \bigvee_a^b(f).$$

Proof. In Theorem 5, let $\epsilon > 0$ and set $v = u + \epsilon$ so as $\epsilon \rightarrow 0^+$ we get the required result. \square

Theorem 6. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be such that f is of bounded variation on $[a, b]$ and g is monotonic nondecreasing on $[a, b]$, then

$$(2.19) \quad |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \leq \frac{1}{4} \left\{ \frac{[g(v) - g(a)] + [g(b) - g(u)]}{2} + \left| \frac{g(v) + g(u)}{2} - \frac{g(a) + g(b)}{2} \right| \right\} \cdot \bigvee_a^b(f),$$

for all $a \leq u < v \leq b$.

Proof. As f is of bounded variation on $[a, b]$ and g is monotonic nondecreasing on $[a, b]$ (which implies that $\Psi_g(t; a, b)$ is absolutely continuous on $[a, b]$), by (2.4) we have

$$(2.20) \quad \begin{aligned} & |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \\ & \leq \frac{1}{v-a} \sup_{r \in [a, v]} \left[(r-a) \left| \frac{1}{r-a} \int_a^r g(s) ds - \frac{1}{v-a} \int_a^b g(s) ds \right| \right] \bigvee_a^v(f) \\ & \quad + \frac{1}{b-u} \sup_{r \in [u, b]} \left[(t-u) \left| \frac{1}{t-u} \int_u^t g(s) ds - \frac{1}{b-u} \int_u^b g(s) ds \right| \right] \bigvee_u^b(f). \end{aligned}$$

Employing the third part of (1.12), setting $d = r, t$ and $c = a, u$, respectively we get

$$(2.21) \quad \left| \frac{1}{r-a} \int_a^r g(s) ds - \frac{1}{v-a} \int_a^v g(s) ds \right| \leq \frac{v-r}{v-a} [g(v) - g(a)].$$

and

$$(2.22) \quad \left| \frac{1}{t-u} \int_u^t g(s) ds - \frac{1}{b-u} \int_u^b g(s) ds \right| \leq \frac{b-t}{b-u} [g(b) - g(u)].$$

Substituting (2.21) and (2.22) in (2.20), we get

$$\begin{aligned} & |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \\ & \leq \frac{1}{(v-a)^2} \sup_{r \in [a, v]} \{(r-a)(v-r)\} \cdot [g(v) - g(a)] \bigvee_a^v(f) \\ & \quad + \frac{1}{(b-u)^2} \sup_{t \in [u, b]} \{(t-u)(b-t)\} \cdot [g(b) - g(u)] \bigvee_u^b(f) \\ & = \frac{1}{4} [g(v) - g(a)] \bigvee_a^v(f) + \frac{1}{4} [g(b) - g(u)] \bigvee_u^b(f) \\ & = \frac{1}{4} \max\{g(v) - g(a), g(b) - g(u)\} \cdot \bigvee_a^b(f) \\ & \leq \frac{1}{4} \left\{ \frac{[g(v) - g(a)] + [g(b) - g(u)]}{2} + \left| \frac{g(v) + g(u)}{2} - \frac{g(a) + g(b)}{2} \right| \right\} \cdot \bigvee_a^b(f), \end{aligned}$$

and thus the proof is finished. \square

Corollary 3. *Under the assumptions of Theorem 6, we have*

$$(2.23) \quad |\mathcal{T}_a^u(f, g) - \mathcal{T}_u^b(f, g)| \leq \left\{ \frac{g(b) - g(a)}{2} + \left| g(u) - \frac{g(a) + g(b)}{2} \right| \right\} \cdot \bigvee_a^b(f),$$

for all $a \leq u \leq b$. In particular case if $u = \frac{a+b}{2}$, then the both inequalities (2.23) gives the same inequality, that is

$$(2.24) \quad \left| \mathcal{T}_a^{\frac{a+b}{2}}(f, g) - \mathcal{T}_{\frac{a+b}{2}}^b(f, g) \right| \leq \left\{ \frac{g(b) - g(a)}{2} + \left| g\left(\frac{a+b}{2}\right) - \frac{g(a) + g(b)}{2} \right| \right\} \cdot \bigvee_a^b(f),$$

Proof. In Theorem 5, let $\epsilon > 0$ and set $v = u + \epsilon$ so as $\epsilon \rightarrow 0^+$ we get the required result. \square

3. BOUNDS FOR LIPSCHITZIAN INTEGRATORS

Theorem 7. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be such that f is L -Lipschitzian on $[a, b]$ and g is an absolutely continuous on $[a, b]$, then*

(3.1)

$$\begin{aligned} & |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \\ & \leq L \begin{cases} \frac{[(b-a)-(v-u)]}{6} \cdot \left[\frac{1}{4} + \left(\frac{\frac{a+b}{2} - \frac{u+v}{2}}{(b-a)-(v-u)} \right)^2 \right] \|g'\|_\infty, & g' \in L_\infty[a, b]; \\ \frac{2[(b-a)-(v-u)]}{(q+1)^{1/q}} \cdot \left[\frac{1}{4} + \left(\frac{\frac{a+b}{2} - \frac{u+v}{2}}{(b-a)-(v-u)} \right)^2 \right] B\left(2, 1 + \frac{1}{q}\right) \cdot \|g'\|_{p, [a, b]}, & g' \in L_p[a, b], \end{cases} \end{aligned}$$

where, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using the fact that for a Riemann integrable function $p : [c, d] \rightarrow \mathbb{R}$ and L -Lipschitzian function $\nu : [c, d] \rightarrow \mathbb{R}$, one has the inequality

$$(3.2) \quad \left| \int_c^d p(t) d\nu(t) \right| \leq L \int_c^d |p(t)| dt.$$

As f is L -Lipschitzian on $[a, b]$, by (3.2) we have

$$\begin{aligned}
 (3.3) \quad & |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \\
 & \leq \frac{L}{v-a} \int_a^v \left| (r-a) \left[\frac{1}{r-a} \int_a^r g(s) ds - \frac{1}{v-a} \int_a^v g(s) ds \right] \right| ds \\
 & \quad + \frac{L}{b-u} \int_u^b \left| (t-u) \left[\frac{1}{t-u} \int_u^t g(s) ds - \frac{1}{b-u} \int_u^b g(s) ds \right] \right| dt \\
 & \leq \frac{1}{2} L \|g'\|_\infty \left[\frac{1}{v-a} \int_a^v (r-a)(v-r) dr + \frac{1}{b-u} \int_u^b (t-a)(b-t) dt \right] \\
 & = \frac{1}{6} L \|g'\|_\infty \left[\frac{(v-a)^2 + (b-u)^2}{2} \right] \\
 & = L \frac{[(b-a) - (v-u)]}{6} \cdot \left[\frac{1}{4} + \left(\frac{\frac{a+b}{2} - \frac{u+v}{2}}{(b-a) - (v-u)} \right)^2 \right] \|g'\|_\infty,
 \end{aligned}$$

where we used the inequality (1.9), with $d = r, t$ and $c = a, u$; respectively.

To obtain the second inequality, setting $d = r, t$ and $c = a, u$; respectively, in (1.10), we get

$$\begin{aligned}
 (3.4) \quad & \left| \frac{1}{r-a} \int_a^r g(s) ds - \frac{1}{v-a} \int_a^v g(s) ds \right| \\
 & \leq \frac{(v-r)^{\frac{1}{q}}}{(q+1)^{1/q} (v-a)^{\frac{1}{q}}} [(r-a)^q + (v-r)^q]^{1/q} \|g'\|_{p,[a,v]}
 \end{aligned}$$

and

$$\begin{aligned}
 (3.5) \quad & \left| \frac{1}{t-u} \int_u^t g(s) ds - \frac{1}{b-u} \int_u^b g(s) ds \right| \\
 & \leq \frac{(b-t)^{\frac{1}{q}}}{(q+1)^{1/q} (b-u)^{\frac{1}{q}}} [(t-u)^q + (b-t)^q]^{1/q} \|g'\|_{p,[u,b]}
 \end{aligned}$$

Substituting (3.4) and (3.5) in (3.3), we get

$$\begin{aligned}
 & |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \\
 & \leq L \frac{\|g'\|_{p,[a,v]}}{(q+1)^{1/q} (v-a)^{1+\frac{1}{q}}} \int_a^v (r-a)(b-r)^{\frac{1}{q}} [(r-a)^q + (v-r)^q]^{1/q} dr \\
 & \quad + L \frac{\|g'\|_{p,[u,b]}}{(q+1)^{1/q} (b-u)^{1+\frac{1}{q}}} \int_u^b (t-u)(b-t)^{\frac{1}{q}} [(t-u)^q + (b-t)^q]^{1/q} dt \\
 & \leq L \frac{\|g'\|_{p,[a,v]}}{(q+1)^{1/q} (v-a)^{1+\frac{1}{q}}} \sup_{r \in [a,v]} [(r-a)^q + (v-r)^q]^{1/q} \int_a^v (r-a)(v-r)^{\frac{1}{q}} dr \\
 & \quad + L \frac{\|g'\|_{p,[u,b]}}{(q+1)^{1/q} (b-u)^{1+\frac{1}{q}}} \sup_{t \in [u,b]} [(t-u)^q + (b-t)^q]^{1/q} \int_u^b (t-u)(b-t)^{\frac{1}{q}} dt
 \end{aligned}$$

$$\begin{aligned}
 &= L \frac{(v-a)^2}{(q+1)^{1/q}} B\left(2, 1 + \frac{1}{q}\right) \cdot \|g'\|_{p,[a,v]} + L \frac{(b-u)^2}{(q+1)^{1/q}} B\left(2, 1 + \frac{1}{q}\right) \cdot \|g'\|_{p,[u,b]} \\
 &\leq L \frac{\|g'\|_{p,[a,b]}}{(q+1)^{1/q}} B\left(2, 1 + \frac{1}{q}\right) \cdot [(v-a)^2 + (b-u)^2] \\
 &= \frac{2[(b-a) - (v-u)]}{(q+1)^{1/q}} \cdot \left[\frac{1}{4} + \left(\frac{\frac{a+b}{2} - \frac{u+v}{2}}{(b-a) - (v-u)} \right)^2 \right] B\left(2, 1 + \frac{1}{q}\right) \cdot \|g'\|_{p,[a,b]}
 \end{aligned}$$

which proves the second inequality in (3.1). \square

Corollary 4. *Under the assumptions of Theorem 7, then*

$$\begin{aligned}
 (3.6) \quad &|\mathcal{T}_a^u(f, g) - \mathcal{T}_u^b(f, g)| \\
 &\leq L \begin{cases} \frac{1}{6} \|g'\|_{\infty} \left[\frac{(u-a)^2 + (b-u)^2}{2} \right], & g' \in L_{\infty}[a, b]; \\ \frac{[(u-a)^2 + (b-u)^2]}{(q+1)^{1/q}} B\left(2, 1 + \frac{1}{q}\right) \cdot \|g'\|_{p,[a,b]}, & g' \in L_p[a, b], \end{cases}
 \end{aligned}$$

where, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. In particular case, if $u = \frac{a+b}{2}$ then

$$\begin{aligned}
 (3.7) \quad &\left| \mathcal{T}_{\frac{a+b}{2}}^{\frac{a+b}{2}}(f, g) - \mathcal{T}_{\frac{a+b}{2}}^b(f, g) \right| \\
 &\leq L \begin{cases} \frac{(b-a)^2}{24} \|g'\|_{\infty}, & g' \in L_{\infty}[a, b]; \\ \frac{(b-a)^2}{2(q+1)^{1/q}} B\left(2, 1 + \frac{1}{q}\right) \cdot \|g'\|_{p,[a,b]}, & g' \in L_p[a, b], \end{cases}
 \end{aligned}$$

Theorem 8. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be such that f is L -Lipschitzian on $[a, b]$ and g is of p -H-Hölder type on $[a, b]$ where $p \in (0, 1]$ and $H > 0$ are given, then*

$$\begin{aligned}
 (3.8) \quad &|\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \\
 &\leq \frac{LH}{(p+1)^2(p+2)} \cdot \left[\frac{(b-a) + (v-u)}{2} + \left| \frac{u+v}{2} - \frac{a+b}{2} \right| \right]^{p+1}.
 \end{aligned}$$

Proof. We repeat the proof of Theorem 7. As f is L -Lipschitzian and g is of p -H-Hölder type on $[a, b]$, by (1.11) we have

$$\begin{aligned}
 &|\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \\
 &\leq \frac{L}{v-a} \int_a^v \left| (r-a) \left[\frac{1}{r-a} \int_a^r g(s) ds - \frac{1}{v-a} \int_a^v g(s) ds \right] \right| dr \\
 &\quad + \frac{L}{b-u} \int_u^b \left| (t-u) \left[\frac{1}{t-u} \int_u^t g(u) du - \frac{1}{b-u} \int_u^b g(u) du \right] \right| dt \\
 &\leq \frac{LH}{(p+1)(v-a)} \int_a^v (r-a)(v-r)^p dr + \frac{LH}{(p+1)(b-u)} \int_u^b (t-u)(b-t)^p dt \\
 &= \frac{LH(v-a)^{p+1}}{(p+1)^2(p+2)} + \frac{LH(b-u)^{p+1}}{(p+1)^2(p+2)}
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{LH}{(p+1)} B(p+1, 2) \cdot [\max\{(v-a), (b-u)\}]^{p+1} \\
&= \frac{LH}{(p+1)^2(p+2)} \cdot \left[\frac{(b-a) + (v-u)}{2} + \left| \frac{u+v}{2} - \frac{a+b}{2} \right| \right]^{p+1},
\end{aligned}$$

where for the last inequality a simple calculation yields that

$$\int_a^b (t-a)(b-t)^p dt = (b-a)^{p+2} \int_0^1 (1-t)t^p dt = \frac{(b-a)^{p+2}}{(p+1)(p+2)},$$

which completes the proof. \square

Corollary 5. *Let f, g be two Lipschitzian mappings on $[a, b]$ with Lipschitz constants $L_f, L_g > 0$, then*

$$(3.9) \quad |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \leq \frac{L_f L_g}{12} \cdot \left[\frac{(b-a) + (v-u)}{2} + \left| \frac{u+v}{2} - \frac{a+b}{2} \right| \right]^2.$$

Moreover,

$$(3.10) \quad |\mathcal{T}_a^u(f, g) - \mathcal{T}_u^b(f, g)| \leq \frac{L_f L_g}{12} \cdot \left[\frac{b-a}{2} + \left| u - \frac{a+b}{2} \right| \right]^2,$$

for all $a \leq u \leq b$. In particular case if $u = \frac{a+b}{2}$, we have

$$(3.11) \quad \left| \mathcal{T}_a^{\frac{a+b}{2}}(f, g) - \mathcal{T}_{\frac{a+b}{2}}^b(f, g) \right| \leq \frac{1}{24} L_f L_g (b-a)^2.$$

Proof. In (3.8), let $p = 1$ we get (3.9). The inequality (3.10) can be obtained by setting $v = u + \epsilon$, $\epsilon > 0$, and letting $\epsilon \rightarrow 0^+$. \square

Theorem 9. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two absolutely continuous on $[a, b]$. If $f' \in L_\alpha[a, b]$, $\alpha, \beta > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then*

$$(3.12) \quad |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \leq \begin{cases} \frac{(v-a)^{\frac{1}{\beta}} + (b-u)^{\frac{1}{\beta}}}{2} \cdot B^{\frac{1}{\beta}}(\beta+1, \beta+1) \cdot \|f'\|_{\alpha, [a, b]} \cdot \|g'\|_{\infty, [a, b]}, & \text{if } g' \in L_\infty[a, b] \\ \frac{(v-a)^{1+\frac{1}{\beta}} + (b-u)^{1+\frac{1}{\beta}}}{(q+1)^{1/q}} \cdot B^{\frac{1}{\beta}}\left(\beta+1, \frac{\beta}{q}+1\right) \|g'\|_{p, [a, b]} \|f'\|_{\alpha, [a, b]}, & \text{if } g' \in L_p[a, b] \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left[(v-a)^{1+\frac{1}{\beta}} + (b-u)^{1+\frac{1}{\beta}} \right] \cdot B^{\frac{1}{\beta}}(\beta+1, \beta+1) \cdot \|g'\|_{1, [a, b]} \|f'\|_{\alpha, [a, b]}, & \text{if } g' \in L_1[a, b] \end{cases}$$

Proof. Taking the absolute value in (1.8) and utilizing the triangle inequality. As $f' \in L_\alpha([a, b])$, by Hölder inequality we have

$$\begin{aligned}
 (3.13) \quad & |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \\
 & \leq \frac{1}{v-a} \int_a^v \left| (r-a) \left[\frac{1}{r-a} \int_a^r g(s) ds - \frac{1}{v-a} \int_a^v g(s) ds \right] \right| |f'(r)| dr \\
 & \quad + \frac{1}{b-u} \int_u^b \left| (t-u) \left[\frac{1}{t-u} \int_u^t g(s) ds - \frac{1}{b-u} \int_u^b g(s) ds \right] \right| |f'(t)| dt \\
 & \leq \frac{1}{v-a} \left(\int_a^v |r-a|^\beta \left| \frac{1}{r-a} \int_a^r g(s) ds - \frac{1}{v-a} \int_a^v g(s) ds \right|^\beta dr \right)^{1/\beta} \\
 & \quad \times \left(\int_a^v |f'(r)|^\alpha dr \right)^{1/\alpha} \\
 & \quad + \frac{1}{b-u} \left(\int_u^b |t-u|^\beta \left| \frac{1}{t-u} \int_u^t g(s) ds - \frac{1}{b-u} \int_u^b g(s) ds \right|^\beta dt \right)^{1/\beta} \\
 & \quad \times \left(\int_u^b |f'(t)|^\alpha dt \right)^{1/\alpha}
 \end{aligned}$$

Now, in (1.9) put $d = r, t$ and $c = a, u$; respectively, then

$$\left| \frac{1}{r-a} \int_a^r g(s) ds - \frac{1}{v-a} \int_a^v g(s) ds \right| \leq \frac{v-r}{2(v-a)} \cdot \|g'\|_{\infty, [a, v]}$$

and

$$\left| \frac{1}{t-u} \int_u^t g(s) du - \frac{1}{b-u} \int_u^b g(s) ds \right| \leq \frac{b-t}{2(b-u)} \cdot \|g'\|_{\infty, [u, b]}$$

Substituting these inequalities in (3.13) we get

$$\begin{aligned}
 & |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \\
 & \leq \frac{1}{2(v-a)^2} \cdot \|f'\|_{\alpha, [a, v]} \cdot \|g'\|_{\infty, [a, v]} \left(\int_a^b (r-a)^\beta (v-r)^\beta dr \right)^{\frac{1}{\beta}} \\
 & \quad + \frac{1}{2(b-u)^2} \cdot \|f'\|_{\alpha, [u, b]} \cdot \|g'\|_{\infty, [u, b]} \left(\int_u^b (t-u)^\beta (b-t)^\beta dt \right)^{\frac{1}{\beta}} \\
 & = \frac{(v-a)^{\frac{1}{\beta}}}{2} \cdot B^{\frac{1}{\beta}}(\beta+1, \beta+1) \cdot \|f'\|_{\alpha, [a, v]} \cdot \|g'\|_{\infty, [a, v]} \\
 & \quad + \frac{(b-u)^{\frac{1}{\beta}}}{2} \cdot B^{\frac{1}{\beta}}(\beta+1, \beta+1) \cdot \|f'\|_{\alpha, [u, b]} \cdot \|g'\|_{\infty, [u, b]} \\
 & \leq \frac{(v-a)^{\frac{1}{\beta}} + (b-u)^{\frac{1}{\beta}}}{2} \cdot B^{\frac{1}{\beta}}(\beta+1, \beta+1) \cdot \|f'\|_{\alpha, [a, b]} \cdot \|g'\|_{\infty, [a, b]}
 \end{aligned}$$

which prove the first inequality in (3.12).

To prove the second and third inequalities in (3.12), we apply (1.10) by setting $d = r, t$ and $c = a, u$; respectively, then we get

$$\begin{aligned}
 & \int_a^v |r-a|^\beta \left| \frac{1}{r-a} \int_a^r g(s) ds - \frac{1}{v-a} \int_a^v g(s) ds \right|^\beta dr \\
 & \leq \begin{cases} \frac{\|g'\|_{p,[a,v]}^\beta}{(q+1)^{\beta/q} (v-a)^{\frac{\beta}{q}}} \int_a^v (r-a)^\beta (v-r)^{\frac{\beta}{q}} [(r-a)^q + (v-r)^q]^{\beta/q} dr, \\ \text{if } g' \in L_p[a, v]; \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{(v-a)^\beta} \|g'\|_{1,[a,v]}^\beta \cdot \int_a^v (r-a)^\beta (v-r)^\beta dr, \text{ if } g' \in L_1[a, v]. \end{cases} \\
 & \leq \begin{cases} \frac{\|g'\|_{p,[a,v]}^\beta}{(q+1)^{\beta/q} (v-a)^{\frac{\beta}{q}}} \sup_{r \in [a,v]} [(r-a)^q + (v-r)^q]^{\beta/q} \int_a^v (r-a)^\beta (v-r)^{\frac{\beta}{q}} dr, \\ \text{if } g' \in L_p[a, v]; \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{(v-a)^\beta} \|g'\|_{1,[a,v]}^\beta \cdot (v-a)^{2\beta+1} B(\beta+1, \beta+1), \text{ if } g' \in L_1[a, v]. \end{cases} \\
 & \quad (3.14) \\
 & = \begin{cases} \frac{(v-a)^{(2+\frac{1}{q})\beta+1}}{(q+1)^{\beta/q} (v-a)^{\frac{\beta}{q}}} \cdot B\left(\beta+1, \frac{\beta}{q}+1\right) \|g'\|_{p,[a,v]}^\beta, & \text{if } g' \in L_p[a, v]; \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{(v-a)^\beta} \|g'\|_{1,[a,v]}^\beta \cdot (v-a)^{2\beta+1} B(\beta+1, \beta+1), & \text{if } g' \in L_1[a, v]. \end{cases}
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 & \int_u^b |t-u|^\beta \left| \frac{1}{t-u} \int_u^t g(s) ds - \frac{1}{b-u} \int_u^b g(s) ds \right|^\beta dt \\
 & \quad (3.15) = \begin{cases} \frac{(b-u)^{(2+\frac{1}{q})\beta+1}}{(q+1)^{\beta/q} (b-u)^{\frac{\beta}{q}}} \cdot B\left(\beta+1, \frac{\beta}{q}+1\right) \|g'\|_{p,[u,b]}^\beta, & \text{if } g' \in L_p[u, b]; \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{(b-u)^\beta} \|g'\|_{1,[u,b]}^\beta \cdot (b-u)^{2\beta+1} B(\beta+1, \beta+1), & \text{if } g' \in L_1[u, b]. \end{cases}
 \end{aligned}$$

Substituting (3.14) and (3.15) in (3.13), we get

$$\begin{aligned}
 & |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \\
 & \leq \begin{cases} \frac{(v-a)^{1+\frac{1}{\beta}} + (b-u)^{1+\frac{1}{\beta}}}{(q+1)^{1/q}} \cdot B^{\frac{1}{\beta}}\left(\beta+1, \frac{\beta}{q}+1\right) \|g'\|_{p,[a,b]} \|f'\|_{\alpha,[a,b]} \\ \left[(v-a)^{1+\frac{1}{\beta}} + (b-u)^{1+\frac{1}{\beta}} \right] \cdot B^{\frac{1}{\beta}}(\beta+1, \beta+1) \cdot \|g'\|_{1,[a,b]} \|f'\|_{\alpha,[a,b]} \end{cases}
 \end{aligned}$$

for all $p, q, \alpha, \beta > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, which proves the second and the third inequalities in (3.12). \square

Corollary 6. *Under the assumptions of Theorem 9, we have*

$$(3.16) \quad |\mathcal{T}_a^u(f, g) - \mathcal{T}_u^b(f, g)| \leq \begin{cases} \frac{(u-a)^{\frac{1}{\beta}} + (b-u)^{\frac{1}{\beta}}}{2} \cdot B^{\frac{1}{\beta}}(\beta+1, \beta+1) \cdot \|f'\|_{\alpha, [a, b]} \cdot \|g'\|_{\infty, [a, b]}, & \text{if } g' \in L_{\infty}[a, b] \\ \frac{(u-a)^{1+\frac{1}{\beta}} + (b-u)^{1+\frac{1}{\beta}}}{(q+1)^{1/q}} \cdot B^{\frac{1}{\beta}}\left(\beta+1, \frac{\beta}{q}+1\right) \|g'\|_{p, [a, b]} \|f'\|_{\alpha, [a, b]}; & \text{if } g' \in L_p[a, b] \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left[(u-a)^{1+\frac{1}{\beta}} + (b-u)^{1+\frac{1}{\beta}}\right] \cdot B^{\frac{1}{\beta}}(\beta+1, \beta+1) \cdot \|g'\|_{1, [a, b]} \|f'\|_{\alpha, [a, b]}, & \text{if } g' \in L_1[a, b] \end{cases}.$$

In particular case, if $u = \frac{a+b}{2}$ we get

$$(3.17) \quad \left| \mathcal{T}_a^{\frac{a+b}{2}}(f, g) - \mathcal{T}_{\frac{a+b}{2}}^b(f, g) \right| \leq \begin{cases} \left(\frac{b-a}{2}\right)^{\frac{1}{\beta}} \cdot B^{\frac{1}{\beta}}(\beta+1, \beta+1) \cdot \|f'\|_{\alpha, [a, b]} \cdot \|g'\|_{\infty, [a, b]}, & \text{if } g' \in L_{\infty}[a, b] \\ \frac{(b-a)^{1+\frac{1}{\beta}}}{2^{1+\frac{1}{\beta}}(q+1)^{\frac{1}{q}}} \cdot B^{\frac{1}{\beta}}\left(\beta+1, \frac{\beta}{q}+1\right) \|g'\|_{p, [a, b]} \|f'\|_{\alpha, [a, b]}; & \text{if } g' \in L_p[a, b] \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left(\frac{b-a}{2}\right)^{1+\frac{1}{\beta}} \cdot B^{\frac{1}{\beta}}(\beta+1, \beta+1) \cdot \|g'\|_{1, [a, b]} \|f'\|_{\alpha, [a, b]}, & \text{if } g' \in L_1[a, b] \end{cases}.$$

Remark 1. *For the second inequality in (3.12) we have the following particular cases:*

(1) *If $\alpha = p$ and $\beta = q$, then we have*

$$(3.18) \quad |\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \leq \frac{(v-a)^{1+\frac{1}{q}} + (b-u)^{1+\frac{1}{q}}}{(q+1)^{1/q}} \cdot B^{\frac{1}{q}}(q+1, 2) \|g'\|_{p, [a, b]} \|f'\|_{p, [a, b]}.$$

Therefore, as $v \rightarrow u^+$ we have

$$(3.19) \quad |\mathcal{T}_a^u(f, g) - \mathcal{T}_u^b(f, g)| \leq \frac{(u-a)^{1+\frac{1}{q}} + (b-u)^{1+\frac{1}{q}}}{(q+1)^{1/q}} \cdot B^{\frac{1}{q}}(q+1, 2) \|g'\|_{p, [a, b]} \|f'\|_{p, [a, b]},$$

and for $u = \frac{a+b}{2}$ we have

$$(3.20) \quad \left| \mathcal{T}_a^{\frac{a+b}{2}}(f, g) - \mathcal{T}_{\frac{a+b}{2}}^b(f, g) \right| \leq \frac{(b-a)^{1+\frac{1}{q}}}{2^{1+\frac{1}{q}}(q+1)^{\frac{1}{q}}} \cdot B^{\frac{1}{q}}(q+1, 2) \|g'\|_{p,[a,b]} \|f'\|_{p,[a,b]}.$$

(2) If $\alpha = q$ and $\beta = p$, then we have

$$(3.21) \quad \left| \mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g) \right| \leq \frac{(v-a)^{1+\frac{1}{p}} + (b-u)^{1+\frac{1}{p}}}{(q+1)^{1/q}} \cdot B^{\frac{1}{p}}\left(p+1, \frac{p}{q}+1\right) \|g'\|_{p,[a,b]} \|f'\|_{q,[a,b]}.$$

Similarly, as $v \rightarrow u^+$, we have

$$(3.22) \quad \left| \mathcal{T}_a^u(f, g) - \mathcal{T}_u^b(f, g) \right| \leq \frac{(u-a)^{1+\frac{1}{p}} + (b-u)^{1+\frac{1}{p}}}{(q+1)^{1/q}} \cdot B^{\frac{1}{p}}\left(p+1, \frac{p}{q}+1\right) \|g'\|_{p,[a,b]} \|f'\|_{q,[a,b]},$$

and for $u = \frac{a+b}{2}$ we have

$$(3.23) \quad \left| \mathcal{T}_a^{\frac{a+b}{2}}(f, g) - \mathcal{T}_{\frac{a+b}{2}}^b(f, g) \right| \leq \frac{(b-a)^{1+\frac{1}{p}}}{2^{1+\frac{1}{p}}(q+1)^{\frac{1}{q}}} \cdot B^{\frac{1}{p}}\left(p+1, \frac{p}{q}+1\right) \|g'\|_{p,[a,b]} \|f'\|_{q,[a,b]},$$

for all $p, q > 1$ with $\frac{1}{p} + \frac{1}{q}$.

Remark 2. In this work, all obtained bounds for the difference between two Čebyšev functional were taken under the assumption that $[a, v] \cap [u, b] = [u, v]$. The same bounds hold with a few changes in the case that $[u, v] \subset [a, b]$. Namely, replace every 'a' (in the obtained results) by 'u'; every 'u' (in the obtained results) by 'a' and accordingly the differences $(v-u)$, $(b-a)$ instead of $(v-a)$, $(b-u)$.

Remark 3. All obtained bounds hold for the Čebyšev functional $|\mathcal{T}_a^b(f, g)|$, this can be done by noting that $|\mathcal{T}_a^v(f, g) - \mathcal{T}_u^b(f, g)| \rightarrow |\mathcal{T}_a^b(f, g)|$ as $v = u \rightarrow a$.

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